A note on E_{11} and three-dimensional gauged supergravity

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ABSTRACT: We determine the gauge symmetries of all p–forms in maximal three-dimensional gauged supergravity ($0 \leq p \leq 3$) by requiring invariance of the Lagrangian. It is shown that in a particular ungauged limit these symmetries are in precise correspondence to those predicted by the very-extended Kac-Moody algebra E_{11}. We demonstrate that whereas in the ungauged limit the bosonic gauge algebra closes off-shell, the closure is only on-shell in the full gauged theory. This underlines the importance of dynamics for understanding the Kac-Moody origin of the symmetries of gauged supergravity.

Contents

1. Introduction

One of the surprising results of supergravity is that the Kaluza-Klein reduction of the maximal 11-dimensional theory on a d-torus yields the exceptional hidden symmetry groups $E_{d(d)}$ for $6 \leq d \leq 9$ [1]. This has led to the conjecture that the over-extended and very-extended Kac-Moody algebras E_{10} [2 – 5] and E_{11} [6 – 12] may be of relevance for the original theory or, more optimistically, be even the ultimate symmetry of M-theory.

Recently, it has been shown that E_{11} (and to some extend also E_{10}) contains information about the possible deformations of supergravity into gauged or massive supergravities [13 – 16]. More precisely, a level decomposition shows that the spectra of E_{11} and E_{10} contain $(D-1)$-form potentials that, via duality, are in precise correspondence with the embedding tensor Θ introduced in [17, 18] for maximal gauged supergravity in $D = 3$ (and subsequently generalized to higher dimensions in [19 – 27]). In addition, the spectrum of E_{11} contains D-form potentials that are in part related to quadratic constraints on the embedding tensor [15, 27].

The embedding tensor approach is based on the introduction of a tensor Θ that is in a particular representation of the duality group and which encodes the gauging. A special feature of the three-dimensional maximally supersymmetric case is that all bosonic matter fields can be dualized to scalars leading to a 128-dimensional $E_{8(8)}/SO(16)$ coset space. However, to gauge a subgroup of the duality group one needs to introduce vectors as well. It was shown in [17, 18] that this can be achieved by a topological term of the form $\Theta A \partial A + \Theta^2 A^3$, where A are the gauge vectors, which in turn do not lead to new degrees of freedom. In higher dimensions, a whole hierarchy of p–form potentials with $0 \leq p \leq D-2$ is introduced [23, 27]. It is a generic feature of this hierarchy that the gauge algebra can be closed off-shell.

For consistency the embedding tensor has to satisfy a set of quadratic constraints. Given a gauged supergravity theory containing the constant embedding tensor one can promote this tensor to an unconstrained scalar field $\Theta(x)$ by adding to the original Lagrangian $\mathcal{L}_g$ a further topological term containing the deformation and top-form potentials as Lagrange multipliers in the following way [15, 27]:

$$\mathcal{L} = \mathcal{L}_g + A_{(D-1)}\partial\Theta + A_{(D)}\Theta\Theta, \tag{1.1}$$

where we have suppressed the duality and space-time indices. These extra potentials complete the hierarchy of potentials to include all p–forms with $0 \leq p \leq D$. There is, however, a subtlety with the bosonic gauge transformations of these new potentials. The gauge-invariance of the original Lagrangian $\mathcal{L}_g$ will be violated by terms proportional to either $\partial\Theta$ or Θ^2. Such terms can always be be canceled by assigning bosonic gauge transformations to the deformation and top-form potentials. However, it is not obvious that the gauge transformations determined like this coincide with those derived from the general formalism, which is valid for the full hierarchy of p–forms in generic dimension. In fact, by inspecting closure of the supersymmetry algebra it has already been pointed out in [27] that the gauge transformations receive modifications when applied to a specific model. Here we are going to derive the full bosonic gauge symmetries for three-dimensional gauged maximal supergravity directly by requiring invariance of the Lagrangian (1.1). In particular, we will find that the closure is only on-shell.

Moreover, we are going to compare the resulting symmetries with those predicted by E_{11}. Since the latter does not give rise to the embedding tensor, but only to its dual deformation potential, naively this would require to take the ungauged limit, i.e. to set the embedding tensor equal to zero.[1] However, we will see that in this limit terms survive in the transformation rules that are not predicted by E_{11}. Instead, we will define a different limit, in which the symmetries precisely match and which, moreover, has the advantage that all p–forms but the top-form survive in the action. We will also see that in this limit the bosonic gauge algebra reduces to an algebra that closes off-shell, in accordance with the level decomposition of E_{11}.

This note is organized as follows. In Section 2 we first introduce the maximal gauged supergravity theory in three dimensions, following [17, 18, 27]. Then we give the complete bosonic gauge transformations of all p–form potentials and show that the bosonic gauge algebra closes on-shell. In the next section we perform the level decomposition of E_{11} and show how the result obtained agrees with a particular limit of the gauged supergravity result discussed in Section 2. In this limit the on-shell closed gauge algebra reduces to an off-shell closed one. Finally, in the conclusions we comment about the consequences of our results for a Kac-Moody approach to gauged supergravity in general.

2. Gauged supergravity in $D = 3$

In this section we give a brief review of gauged maximal supergravity in $D = 3$ [17, 18, 27].

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[1]Recently, a scheme has been proposed to include the embedding tensor via a further extension of E_{11} [28]. Here we will not explore this possibility.

In the first subsection we will introduce the Lagrangian and the embedding tensor. In the following subsection we will introduce an equivalent formulation [27], in which the non-propagating 2-form and 3-form fields predicted by E_{11} appear, and determine their bosonic gauge symmetries.

2.1 The Lagrangian and the embedding tensor

The propagating bosonic degrees of freedom of maximal supergravity in $D = 3$ consist of 128 scalar fields parameterizing the coset space $E_{8(8)}/SO(16)$. Besides, there are the topological metric and, in gauged supergravity, Chern-Simons vectors. The 128 scalars are encoded in the $E_{8(8)}$ valued matrix $\mathcal{V}^{\mathcal{M}}{}_{\mathcal{A}}$, where $\mathcal{M}, \mathcal{A}, \ldots = 1, \ldots, 248$ denote adjoint indices of $E_{8(8)}$. We indicate by letters from the middle and the beginning of the alphabet 'curved' indices corresponding the global left action and 'flat' indices corresponding to the local right action, respectively. The scalars enter the Lagrangian via the non-compact part of the Maurer-Cartan forms

$$\mathcal{V}^{-1}D_\mu \mathcal{V} \;=\; \frac{1}{2}Q_\mu^{IJ} X^{IJ} + P_\mu^A Y^A \;, \tag{2.1}$$

which we wrote according to the $SO(16)$ decomposition $\mathbf{248} = \mathbf{120} \oplus \mathbf{128}$. Here X^{IJ} denote the $SO(16)$ generators, with vector indices $I, J, \ldots = 1, \ldots, 16$, and Y^A are the non-compact generators transforming as spinors under $SO(16)$, i.e. with spinor indices $A, B, \ldots = 1, \ldots, 128$.[2]

In order for the Maurer-Cartan forms to be invariant under the local transformations

$$\delta\mathcal{V} \;=\; \hat{g}(x)\mathcal{V} \;, \qquad \hat{g} \in \mathfrak{g}_0 \subset \mathfrak{e}_{8(8)} \;, \tag{2.2}$$

we introduced a gauge-covariant derivative,

$$\mathcal{V}^{-1}D_\mu \mathcal{V} \;=\; \mathcal{V}^{-1}\partial_\mu \mathcal{V} - gA_\mu{}^{\mathcal{M}}\Theta_{\mathcal{M}\mathcal{N}}(\mathcal{V}^{-1}t^{\mathcal{N}}\mathcal{V}) \;, \tag{2.3}$$

where g is the gauge coupling constant. The symmetric tensor $\Theta_{\mathcal{M}\mathcal{N}}$ is the embedding tensor, which encodes the embedding of the gauge group G_0 into the global symmetry group $E_{8(8)}$. More precisely, the gauge algebra $\mathfrak{g}_0$ is spanned by

$$X_{\mathcal{M}} \;=\; \Theta_{\mathcal{M}\mathcal{N}}t^{\mathcal{N}} \;, \tag{2.4}$$

in which $t^{\mathcal{M}}$ denote the global $\mathfrak{e}_{8(8)}$ symmetry generators with structure constants $f^{\mathcal{M}\mathcal{N}}{}_{\mathcal{K}}$. In particular, the dimension of $\mathfrak{g}_0$ is given by the rank of $\Theta_{\mathcal{M}\mathcal{N}}$. In this formalism, the gauging takes a fully $E_{8(8)}$ covariant form, since all indices are $E_{8(8)}$ indices. Nevertheless, the duality group is no longer a symmetry due to the fact that the constant Θ cannot transform under $E_{8(8)}$. Rather, it acts as a projector, which breaks the symmetry down to the gauge group G_0 in (2.4).[3]

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[2]Our $E_{8(8)}$ conventions are as in [18]. For other decompositions of $E_{8(8)}$ and their application to maximal gauged supergravity see [29–31].

[3]Alternatively, one could say that $E_{8(8)}$ transforms one theory into another theory with different values of the constant Θ.

The gauged supergravity is described by the Lagrangian

$$\mathcal{L}_g = -\frac{1}{4}eR + \frac{1}{4}eP^{\mu A}P_{\mu A} - eV$$
$$- \frac{1}{4}g\varepsilon^{\mu\nu\rho}A_\mu{}^{\mathcal{M}}\Theta_{\mathcal{MN}}\left(\partial_\nu A_\rho{}^{\mathcal{N}} - \frac{1}{3}g\Theta_{\mathcal{KS}}f^{\mathcal{NS}}{}_{\mathcal{L}}A_\nu{}^{\mathcal{K}}A_\rho{}^{\mathcal{L}}\right) , \tag{2.5}$$

where we ignored the fermionic terms. The scalar potential V is completely determined by Θ via the so-called T-tensor,[4]

$$T_{\mathcal{A}|\mathcal{B}} = \mathcal{V}^{\mathcal{M}}{}_{\mathcal{A}}\mathcal{V}^{\mathcal{N}}{}_{\mathcal{B}}\Theta_{\mathcal{MN}} . \tag{2.6}$$

Explicitly, one has

$$V = -\frac{1}{8}g^2\left(A_1^{IJ}A_1^{IJ} - \frac{1}{2}A_2^{I\dot{A}}A_2^{I\dot{A}}\right) , \tag{2.7}$$

where

$$A_1^{IJ} = \frac{8}{7}\theta\delta^{IJ} + \frac{1}{7}T_{IK|JK} , \qquad A_2^{I\dot{A}} = -\frac{1}{7}\Gamma^J_{A\dot{A}}T_{IJ|A} . \tag{2.8}$$

Here $\theta \equiv \frac{1}{248}\eta^{\mathcal{MN}}\Theta_{\mathcal{MN}} = \frac{1}{248}\eta^{\mathcal{AB}}T_{\mathcal{A}|\mathcal{B}}$ with the Cartan-Killing metric $\eta^{\mathcal{MN}}$. The particular combinations A_1 and A_2 in (2.8) also enter the supersymmetry variations of the fermions [18]. In the following we give a reformulation of the scalar potential in terms of the $E_{8(8)}$ matrix $G^{\mathcal{MN}} = \mathcal{V}^{\mathcal{M}}{}_{\mathcal{A}}\mathcal{V}^{\mathcal{N}}{}_{\mathcal{B}}\delta^{\mathcal{AB}}$. Using the inverse of the relations (2.8) [18], we find[5]

$$V = \frac{1}{32}g^2 G^{\mathcal{MN},\mathcal{KL}}\Theta_{\mathcal{MN}}\Theta_{\mathcal{KL}} , \tag{2.9}$$

where

$$G^{\mathcal{MN},\mathcal{KL}} = \frac{1}{14}G^{\mathcal{MK}}G^{\mathcal{NL}} + G^{\mathcal{MK}}\eta^{\mathcal{NL}} - \frac{3}{14}\eta^{\mathcal{MK}}\eta^{\mathcal{NL}} - \frac{4}{6727}\eta^{\mathcal{MN}}\eta^{\mathcal{KL}} . \tag{2.10}$$

Note that the Chern-Simons term in (2.5) has the effect that varying with respect to the gauge fields $A_\mu{}^{\mathcal{M}}$ one obtains a duality relation between the vectors and scalars,

$$e^{-1}\varepsilon^{\mu\nu\rho}\Theta_{\mathcal{MN}}F_{\nu\rho}{}^{\mathcal{N}} = -2\Theta_{\mathcal{MN}}\mathcal{V}^{\mathcal{N}}{}_A P^{\mu A} \equiv -2\Theta_{\mathcal{MN}}J^{\mu\mathcal{N}} . \tag{2.11}$$

Here we introduced the current $J_\mu{}^{\mathcal{M}}$, which in the ungauged theory is the Noether current corresponding to the global $E_{8(8)}$ symmetry. However, in the gauged theory this symmetry is broken, and therefore the covariant conservation is violated by terms of order $\mathcal{O}(g)$ induced by the scalar potential,

$$D_\mu\left(eJ^{\mu\mathcal{M}}\right) = \mathcal{O}(g) . \tag{2.12}$$

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[4]Following [27] we use a vertical bar to distinguish between the two indices of T.

[5]For performing the required gamma matrix calculations we used the Mathematica package GAMMA [32].

We emphasize that (2.11) is not the 'naive' duality relation in that both sides appear projected by the embedding tensor. Consequently, only those vector fields participating in the gauging enter (2.11), which therefore cannot be used to eliminate the full 248 vector fields in terms of the scalars. As has been noted in [27], there is one 'extra' gauge symmetry related to the duality relation,

$$\delta_\chi A_\mu{}^{\mathcal{M}} \;=\; \xi_\chi^\nu \big(F_{\mu\nu}{}^{\mathcal{M}} + \tilde{J}_{\mu\nu}{}^{\mathcal{M}} \big) \,, \tag{2.13}$$

where we defined the Hodge dual $\tilde{J}_{\mu\nu}{}^{\mathcal{M}} = e\varepsilon_{\mu\nu\rho}J^{\rho\mathcal{M}}$ of the current in (2.11). Due to the missing contraction with $\Theta_{\mathcal{M}\mathcal{N}}$, this is not an equations-of-motion symmetry, but nevertheless leaves the action invariant. Though (2.13) seems to be necessary for closure of the supersymmetry algebra [27], we will not encounter this symmetry any further in this paper.

The Lagrangian (2.5) is invariant under the gauge transformations (2.2) and the following gauge transformations of the vector potentials

$$\delta A_\mu{}^{\mathcal{M}} \;=\; D_\mu \Lambda^{\mathcal{M}} \;\equiv\; \partial_\mu \Lambda^{\mathcal{M}} - g f^{\mathcal{M}\mathcal{N}}{}_{\mathcal{K}} \Theta_{\mathcal{N}\mathcal{L}} A_\mu{}^{\mathcal{L}} \Lambda^{\mathcal{K}} \,, \tag{2.14}$$

where the gauge parameter is related to the transformation (2.2) via $\hat{g} = g\Lambda^{\mathcal{M}}\Theta_{\mathcal{M}\mathcal{N}}t^{\mathcal{N}}$. Even though (2.14) seems to describe a 248-dimensional local symmetry, it is actually more subtle, since the gauge vectors $A_\mu{}^{\mathcal{M}}$ and their variations appear in the Lagrangian always contracted with the embedding tensor, which in turn reduces the number of independent vector fields to $\dim G_0 = \text{rank}(\Theta)$. Moreover, the embedding tensor has to satisfy a number of constraints in order for the action to be invariant under the various symmetries. First of all, consistency with local supersymmetry implies a linear constraint on $\Theta_{\mathcal{M}\mathcal{N}}$: a priori it takes values in the symmetric tensor product

$$(\mathbf{248} \otimes \mathbf{248})_{\text{sym}} \;=\; \underline{\mathbf{1}} \oplus \underline{\mathbf{3875}} \oplus \mathbf{27000} \,, \tag{2.15}$$

but supersymmetry requires that only the underlined representations appear. Note that the singlet component of the embedding tensor corresponds to a gauging of the full $E_{8(8)}$ duality group. In the following we will denote symmetrization in two adjoint indices $\mathcal{M},\mathcal{N}$ and subsequent projecting away the $\mathbf{27000}$ representation by $\langle \mathcal{M}\mathcal{N} \rangle$, e.g.

$$\Theta_{\mathcal{M}\mathcal{N}} = \Theta_{\langle \mathcal{M}\mathcal{N} \rangle} \,, \tag{2.16}$$

where the explicit form of the projector has been determined in [34].

Secondly, invariance of the embedding tensor (and thus gauge invariance of the action (2.5) under (2.14)), requires the quadratic constraint [18]

$$\mathcal{Q}_{\mathcal{M}\mathcal{N},\mathcal{P}} \;\equiv\; \Theta_{\mathcal{K}\mathcal{P}}\Theta_{\mathcal{L}(\mathcal{M}}f^{\mathcal{K}\mathcal{L}}{}_{\mathcal{N})} \;=\; 0 \,. \tag{2.17}$$

From this definition one infers that the quadratic constraint satisfies

$$\mathcal{Q}_{(\mathcal{M}\mathcal{N},\mathcal{P})} \;=\; 0 \,, \qquad \eta^{\mathcal{M}\mathcal{N}}\mathcal{Q}_{\mathcal{M}\mathcal{N},\mathcal{P}} \;=\; 0 \,. \tag{2.18}$$

Note that for $GL(n)$ groups the first condition would imply that $\mathcal{Q}$ lives in an irreducible representation.[6] However, this does not hold for $E_{8(8)}$, and the representation content of (2.17) can be analyzed as follows [27]. Due to the linear constraint on Θ, the symmetric indices of $\mathcal{Q}_{\mathcal{MN},\mathcal{P}}$ will be in $\mathbf{3875}$, where the absence of the singlet follows by the second equation in (2.18). Naively the quadratic constraint (2.17) takes therefore values in

$$\mathbf{3875} \otimes \mathbf{248} \;=\; \mathbf{248} \oplus \underline{\mathbf{3875}} \oplus \mathbf{30380} \oplus \underline{\mathbf{147250}} \oplus \mathbf{779247} \;. \tag{2.19}$$

However, the first condition in (2.18) implies that all representations contained in the totally symmetric tensor product $(\mathbf{248} \otimes \mathbf{248} \otimes \mathbf{248})_{\mathrm{sym}}$ will be absent. This in turn reduces the irreducible representations of $\mathcal{Q}_{\mathcal{MN},\mathcal{P}}$ to those underlined in (2.19). By abuse of notation we will denote the projector onto these representations also by brackets $\langle \, \rangle$, but note that its explicit form is not required for our analysis.

2.2 Deformation and top-form potentials

We will now present an equivalent reformulation of the gauged supergravity Lagrangian (2.5), in which so-called deformation and top-form potentials appear. This turns out to be necessary in order to match the spectrum predicted by E_{11}. Formally, this can be understood as follows. As we noted above, the gauged supergravity is not invariant under $E_{8(8)}$, since as 'coupling constants', the $\Theta_{\mathcal{MN}}$ do not transform under the duality group. Promoting the embedding tensor to a dynamical, i.e. space-time dependent field $\Theta_{\mathcal{MN}}(x)$, such that it transforms under global rotations according to its index structure, gives back the full $E_{8(8)}$ invariance. However, this violates the supersymmetry and gauge invariance by terms proportional to $\partial_\mu \Theta_{\mathcal{MN}}$. This can be compensated by adding a 2-form potential to the action, and by assigning appropriate supersymmetry and gauge variations to it. Moreover, the quadratic constraint (2.17) on $\Theta_{\mathcal{MN}}$ can be implemented on-shell by means of a Lagrange multiplier term containing a top-form (3-form) potential. In total we extend the action to [27, 15]

$$\mathcal{L}_{\mathrm{tot}} \;=\; \mathcal{L}_g + \frac{1}{4} g \varepsilon^{\mu\nu\rho} D_\mu \Theta_{\mathcal{MN}} B_{\nu\rho}{}^{\mathcal{MN}} - \frac{1}{6} g^2 \Theta_{\mathcal{KP}} \Theta_{\mathcal{L}(\mathcal{M}} f^{\mathcal{KL}}{}_{\mathcal{N})} \varepsilon^{\mu\nu\rho} C_{\mu\nu\rho}{}^{\mathcal{MN},\mathcal{P}} \;, \tag{2.20}$$

where the embedding tensor now satisfies only the linear constraint. Consequently, the deformation potential takes values in $\mathbf{1} \oplus \mathbf{3875}$, while the top-form lives in $\mathbf{3875} \oplus \mathbf{147250}$, in accordance with (2.19). We have defined a formal covariant derivative $D_\mu \Theta_{\mathcal{MN}}$ as

$$D_\mu \Theta_{\mathcal{MN}} \;=\; \partial_\mu \Theta_{\mathcal{MN}} - 2g A_\mu{}^{\mathcal{P}} \Theta_{\mathcal{KP}} \Theta_{\mathcal{L}(\mathcal{M}} f^{\mathcal{KL}}{}_{\mathcal{N})} \;. \tag{2.21}$$

The combination $D_\mu \Theta_{\mathcal{MN}}$ is strictly speaking not a covariant derivative. It would be the covariant derivative if $\Theta_{\mathcal{MN}}$ would transform under the gauge group according to its index structure. However, it is convenient to set up the calculation using a basis of gauge transformations in which the embedding tensor is gauge-invariant, $\delta_\Lambda \Theta_{\mathcal{MN}} = 0$. This can always be achieved by redefining the gauge transformations with an extra equation of motion symmetry involving the embedding tensor and the top-form potential. In fact, the coefficient

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[6]The projector which implements this condition reads $X^{\langle \mathcal{MN},\mathcal{P} \rangle} = \frac{2}{3}(X^{(\mathcal{MN}),\mathcal{P}} - X^{\mathcal{P}(\mathcal{M},\mathcal{N})})$.

of the A term in (2.21) can be arbitrarily changed by a redefinition of the top-form potential in which the 3-form $C_{\mu\nu\rho}{}^{\mathcal{MN},\mathcal{P}}$ is shifted by terms proportional to $B_{[\mu\nu}{}^{\langle\mathcal{MN}}A_{\rho]}{}^{\mathcal{P}\rangle}$. In general, there are several equivalent ways to present the gauge transformations that are all related via redefinitions of fields/parameters and/or adding further equations of motion symmetries. This will be of relevance when comparing our results with the ones predicted by E_{11}, see the next section. Note that the equations of motion of $B_{\mu\nu}{}^{\mathcal{MN}}$ and $C_{\mu\nu\rho}{}^{\mathcal{MN},\mathcal{P}}$ give back the constancy of $\Theta_{\mathcal{MN}}$ and the quadratic constraints.

Using a particular choice of basis we now wish to determine the gauge transformations of B and C, which are required for the gauge invariance of the action (2.20). (For their supersymmetry transformations see [27].) First of all, the Chern-Simons term varies as

$$
\begin{aligned}
\delta_\Lambda \mathcal{L}_{\rm CS} = & -\frac{1}{4}g\varepsilon^{\mu\nu\rho}D_\mu\Theta_{\mathcal{MN}}D_\nu\Lambda^{\mathcal{M}}A_\rho{}^{\mathcal{N}} \\
& +\frac{1}{6}g^2\varepsilon^{\mu\nu\rho}\Theta_{\mathcal{KP}}\Theta_{\mathcal{L}(\mathcal{M}}f^{\mathcal{KL}}{}_{\mathcal{N})}A_\mu{}^{\mathcal{P}}A_\nu{}^{\mathcal{M}}D_\rho\Lambda^{\mathcal{N}} \;.
\end{aligned}
\tag{2.22}
$$

Also the scalar-kinetic term is no longer gauge-invariant, since the P_μ^A vary according to

$$
\delta_\Lambda P_\mu^A = gD_\mu\Theta_{\mathcal{MN}}\Lambda^{\mathcal{M}}\mathcal{V}^{\mathcal{N}A} \;.
\tag{2.23}
$$

In addition we have to remember the variation of $A_\mu{}^{\mathcal{M}}$ inside the derivative $D_\mu\Theta_{\mathcal{MN}}$. This gives a contribution proportional to $D\Lambda B$ and the quadratic constraint and can therefore be canceled by an extra variation of the top-form. Finally, the T-tensor transforms as

$$
\delta T_{\mathcal{A}|\mathcal{B}} \;=\; -2g\mathcal{Q}_{\mathcal{MN},\mathcal{P}}\mathcal{V}^{\mathcal{M}}{}_{\mathcal{A}}\mathcal{V}^{\mathcal{N}}{}_{\mathcal{B}}\Lambda^{\mathcal{P}} \;,
\tag{2.24}
$$

and, consequently, the scalar potential varies into the quadratic constraint. Collecting these terms, the non-invariance of the Lagrangian can be compensated by introducing the following transformation rules

$$
\begin{aligned}
\delta B_{\mu\nu}{}^{\mathcal{MN}} = & \; D_{[\mu}\Lambda^{\langle\mathcal{M}}A_{\nu]}{}^{\mathcal{N}\rangle} - \Lambda^{\langle\mathcal{M}}\tilde{J}_{\mu\nu}{}^{\mathcal{N}\rangle} \;, \\
\delta C_{\mu\nu\rho}{}^{\mathcal{MN},\mathcal{P}} = & \; -3D_{[\mu}\Lambda^{\langle\mathcal{P}}B_{\nu\rho]}{}^{\mathcal{MN}\rangle} + A_{[\mu}{}^{\langle\mathcal{P}}A_\nu{}^{\mathcal{M}}D_{\rho]}\Lambda^{\mathcal{N}\rangle} \\
& +\frac{1}{16}ge\varepsilon_{\mu\nu\rho}\Lambda^{\langle\mathcal{P}}\left(-\frac{1}{7}G^{\mathcal{M}|\mathcal{K}|}G^{\mathcal{N}\rangle\mathcal{L}} - G^{\mathcal{M}|\mathcal{K}|}\eta^{\mathcal{N}\rangle\mathcal{L}}\right)\Theta_{\mathcal{KL}} \;.
\end{aligned}
\tag{2.25}
$$

At this point let us note again that the explicit form of the projectors indicated in (2.25) is not required, since in the variation of the Lagrangian these terms are always multiplied by $\partial_\mu\Theta_{\mathcal{MN}}$ or the quadratic constraint, and so their projection is manifest.

Next we are going to determine the gauge variations of B and C under their own parameter, Λ_μ and $\Lambda_{\mu\nu}$, respectively. We first consider the gauge transformations with parameter Λ_μ. Defining $\delta B_{\mu\nu}{}^{\mathcal{MN}} = D_{[\mu}\Lambda_{\nu]}{}^{\mathcal{MN}}$ does not leave (2.20) invariant, since the

'covariant' derivatives D_μ do not commute.[7] Rather one finds the variation

$$\delta\left(\frac{1}{4}g\varepsilon^{\mu\nu\rho}D_\mu\Theta_{\mathcal{MN}}B_{\nu\rho}{}^{\mathcal{MN}}\right) = \frac{1}{8}g\varepsilon^{\mu\nu\rho}\Lambda_\mu{}^{\mathcal{MN}}[D_\nu, D_\rho]\Theta_{\mathcal{MN}} \tag{2.26}$$

$$= -\frac{1}{4}g^2\varepsilon^{\mu\nu\rho}\Lambda_\mu{}^{\mathcal{MN}}F_{\nu\rho}{}^{\mathcal{P}}\Theta_{\mathcal{KP}}\Theta_{\mathcal{L(M}}f^{\mathcal{KL}}{}_{\mathcal{N})}$$

$$+\frac{1}{2}g^2\varepsilon^{\mu\nu\rho}\Lambda_\mu{}^{\mathcal{MN}}A_\nu{}^{\mathcal{P}}(D_\rho\Theta_{\mathcal{KP}})\Theta_{\mathcal{L(M}}f^{\mathcal{KL}}{}_{\mathcal{N})}$$

$$+\frac{1}{2}g^3\varepsilon^{\mu\nu\rho}\Theta_{\mathcal{PQ}}\Theta_{\mathcal{R(K}}f^{\mathcal{QR}}{}_{\mathcal{L})}\Theta_{\mathcal{SM}}f^{\mathcal{SL}}{}_{\mathcal{N}}\Lambda_\mu{}^{\mathcal{MN}}A_\nu{}^{\mathcal{P}}A_\rho{}^{\mathcal{K}}\,.$$

To compensate these we add a Stückelberg like shift transformation to the gauge vectors, $\delta'A_\mu{}^{\mathcal{M}} = -g\Theta_{\mathcal{N(K}}f^{\mathcal{MN}}{}_{\mathcal{L})}\Lambda_\mu{}^{\mathcal{KL}}$. The Chern-Simons term then picks up an additional variation, which precisely cancels the variation in (2.26) proportional to the field strength. Apart from that, the P_μ^A vary as

$$\delta P_\mu^A = g^2\Theta_{\mathcal{MN}}\Theta_{\mathcal{P(K}}f^{\mathcal{MP}}{}_{\mathcal{L})}\Lambda_\mu{}^{\mathcal{KL}}\mathcal{V}^{\mathcal{N}A}\,, \tag{2.27}$$

while the variation of $A_\mu{}^{\mathcal{M}}$ inside the derivative $D_\mu\Theta_{\mathcal{MN}}$ also gives rise to a term proportional to the quadratic constraint, which both can be absorbed into an extra transformation of C.

We next consider the gauge symmetry of the top-form, $\delta C_{\mu\nu\rho}{}^{\mathcal{MN},\mathcal{K}} = D_{[\mu}\Lambda_{\nu\rho]}{}^{\mathcal{MN},\mathcal{P}}$. The action transforms into a total derivative and terms proportional to $D_\mu\Theta_{\mathcal{MN}}$. The latter can be compensated by a shift transformation of B under $\Lambda_{\mu\nu}$. This establishes the gauge-invariance of the action with respect to $\Lambda_{\mu\nu}$.

Summarizing, we have shown that the bosonic gauge transformations that leave the action corresponding to the Lagrangian (2.20) invariant are given by

$$\delta A_\mu{}^{\mathcal{M}} = D_\mu\Lambda^{\mathcal{M}} - g\Theta_{\mathcal{NK}}f^{\mathcal{MN}}{}_{\mathcal{L}}\Lambda_\mu{}^{\mathcal{KL}}\,, \tag{2.28}$$

$$\delta B_{\mu\nu}{}^{\mathcal{MN}} = D_{[\mu}\Lambda_{\nu]}{}^{\mathcal{MN}} + \delta A_{[\mu}{}^{\langle\mathcal{M}}A_{\nu]}{}^{\mathcal{N}\rangle} - \Lambda^{\langle\mathcal{M}}\tilde{J}_{\mu\nu}{}^{\mathcal{N}\rangle}$$

$$+\frac{2}{3}g\Theta_{\mathcal{KL}}f^{\mathcal{K}\langle\mathcal{M}}{}_{\mathcal{P}}\left(\Lambda_{\mu\nu}{}^{|\mathcal{LP}|,\mathcal{N}\rangle} - \Lambda_{\mu\nu}{}^{\mathcal{N}\rangle\mathcal{P},\mathcal{L}}\right)\,,$$

$$\delta C_{\mu\nu\rho}{}^{\mathcal{MN},\mathcal{P}} = D_{[\mu}\Lambda_{\nu\rho]}{}^{\mathcal{MN},\mathcal{P}} - 3\,\delta A_{[\mu}{}^{\langle\mathcal{P}}B_{\nu\rho]}{}^{\mathcal{MN}\rangle} + A_{[\mu}{}^{\langle\mathcal{P}}A_\nu{}^{\mathcal{M}}\delta A_{\rho]}{}^{\mathcal{N}\rangle}$$

$$+\frac{3}{2}\Lambda_{[\mu}{}^{\langle\mathcal{MN}}\tilde{J}_{\nu\rho]}{}^{\mathcal{P}\rangle} + \frac{1}{16}ge\varepsilon_{\mu\nu\rho}\Lambda^{\langle\mathcal{P}}\left(-\frac{1}{7}G^{\mathcal{M}|\mathcal{K}|}G^{\mathcal{N}\rangle\mathcal{L}} - G^{\mathcal{M}|\mathcal{K}|}\eta^{\mathcal{N}\rangle\mathcal{L}}\right)\Theta_{\mathcal{KL}}\,.$$

As a consistency check we verify the closure of the gauge algebra. We first consider the $[\mathbf{1}, \mathbf{1}]$ commutator. Here we indicate the generators associated to the corresponding p–forms with $\mathbf{1}$, $\mathbf{2}$ and $\mathbf{3}$ and their gauge variation with $\delta^{(p)}$. We find

$$\left[\delta_\Lambda^{(1)}, \delta_\Sigma^{(1)}\right]A_\mu{}^{\mathcal{M}} = \left(\delta_{\tilde{\Lambda}}^{(1)} + \delta_{\tilde{\Lambda}}^{(2)} + \delta_{\tilde{\Lambda}}^{(3)}\right)A_\mu{}^{\mathcal{M}}\,, \tag{2.29}$$

$$\left[\delta_\Lambda^{(1)}, \delta_\Sigma^{(1)}\right]B_{\mu\nu}{}^{\mathcal{MN}} = \left(\delta_{\tilde{\Lambda}}^{(1)} + \delta_{\tilde{\Lambda}}^{(2)} + \delta_{\tilde{\Lambda}}^{(3)}\right)B_{\mu\nu}{}^{\mathcal{MN}}$$

$$+gf^{\mathcal{MK}}{}_{\mathcal{P}}\Theta_{\mathcal{KL}}\left(F_{\mu\nu}{}^{\mathcal{L}} + \tilde{J}_{\mu\nu}{}^{\mathcal{L}}\right)\Lambda^{[\mathcal{P}}\Sigma^{\mathcal{N}]}\,,$$

$$\left[\delta_\Lambda^{(1)}, \delta_\Sigma^{(1)}\right]C_{\mu\nu\rho}{}^{\mathcal{MN},\mathcal{P}} = \left(\delta_{\tilde{\Lambda}}^{(1)} + \delta_{\tilde{\Lambda}}^{(2)} + \delta_{\tilde{\Lambda}}^{(3)}\right)C_{\mu\nu\rho}{}^{\mathcal{MN},\mathcal{P}}\,,$$

---

[7]It turns out that using the derivative D_μ in this expression corresponds to a particular choice of basis for the parameter $\Lambda_{\mu\nu}$.

where the transformation parameters are given by

$$\tilde{\Lambda}^{\mathcal{M}} = -g\Theta_{\mathcal{N}\mathcal{K}}f^{\mathcal{M}\mathcal{N}}{}_{\mathcal{L}}\Lambda^{[\mathcal{K}}\Sigma^{\mathcal{L}]} \,, \tag{2.30}$$
$$\tilde{\Lambda}_{\mu}{}^{\mathcal{M}\mathcal{N}} = D_{\mu}\Lambda^{\langle\mathcal{M}}\Sigma^{\mathcal{N}\rangle} - D_{\mu}\Sigma^{\langle\mathcal{M}}\Lambda^{\mathcal{N}\rangle} \,,$$
$$\tilde{\Lambda}_{\mu\nu}{}^{\mathcal{M}\mathcal{N},\mathcal{P}} = 3\Lambda^{\langle\mathcal{M}}\tilde{J}_{\mu\nu}{}^{\mathcal{N}}\Sigma^{\mathcal{P}\rangle} \,.$$

We note that in deriving (2.29) we have made use of the scalar equations of motion, the constancy of the embedding tensor and the quadratic constraint, i.e. the closure is only on-shell. For simplicity, we do not give these terms explicitly in the above expressions, but just indicate that the on-shell closure on the deformation potential is guaranteed by the duality relation (2.11) between vectors and scalars. One may wonder whether it is possible to close this algebra off-shell by using the extra symmetries discussed in [27] (see eq. (2.13)). However, on the deformation potential they act as $\delta_{\chi}B_{\mu\nu}{}^{\mathcal{M}\mathcal{N}} \sim A_{\mu}{}^{\mathcal{M}}\delta_{\chi}A_{\nu}{}^{\mathcal{N}}$ and are therefore not of the form required by (2.29) — apart from the fact that it would still not be clear how to eliminate the other equations of motion. We conclude that there is no straightforward way to achieve an off-shell closure, though the possibility of introducing auxiliary fields, etc., might be worth to investigate.

The only other non-trivial commutator to consider is $[\mathbf{1}, \mathbf{2}]$. We find, for instance,

$$\left[\delta_{\Lambda}^{(1)}, \delta_{\Sigma}^{(2)}\right]B_{\mu\nu}{}^{\mathcal{M}\mathcal{N}} = \delta_{\tilde{\Sigma}}^{(3)}B_{\mu\nu}{}^{\mathcal{M}\mathcal{N}} \,, \tag{2.31}$$

where

$$\tilde{\Sigma}_{\mu\nu}{}^{\mathcal{M}\mathcal{N},\mathcal{P}} = 3\Sigma_{[\mu}{}^{\langle\mathcal{M}\mathcal{N}}D_{\nu]}\Lambda^{\mathcal{P}\rangle} \,. \tag{2.32}$$

This concludes our discussion of the commutator algebra.

We end this section by considering the duality relation between the deformation potential and the embedding tensor. Varying the action corresponding to (2.20) with respect to $\Theta_{\mathcal{M}\mathcal{N}}$ yields the following 'duality relation':

$$e^{-1}\varepsilon^{\mu\nu\rho}G_{\mu\nu\rho}{}^{\mathcal{M}\mathcal{N}} + 2A_{\mu}{}^{\langle\mathcal{M}}J^{\mu\mathcal{N}\rangle} = \frac{1}{4}gG^{\mathcal{M}\mathcal{N},\mathcal{K}\mathcal{L}}\Theta_{\mathcal{K}\mathcal{L}} \,. \tag{2.33}$$

Here we have defined

$$G_{\mu\nu\rho}{}^{\mathcal{M}\mathcal{N}} = D_{[\mu}B_{\nu\rho]}{}^{\mathcal{M}\mathcal{N}} + A_{[\mu}{}^{\langle\mathcal{M}}\partial_{\nu}A_{\rho]}{}^{\mathcal{N}\rangle} - 2g\Theta_{\mathcal{K}\mathcal{L}}f^{\mathcal{K}\langle\mathcal{M}}{}_{\mathcal{P}}A_{[\mu}{}^{\mathcal{N}\rangle}B_{\nu\rho]}{}^{\mathcal{L}\mathcal{P}}$$
$$- \frac{2}{3}g\Theta_{\mathcal{K}\mathcal{L}}f^{\mathcal{K}\langle\mathcal{M}}{}_{\mathcal{P}}\left(C_{\mu\nu\rho}{}^{|\mathcal{L}\mathcal{P}|,\mathcal{N}\rangle} - C_{\mu\nu\rho}{}^{\mathcal{N}\rangle\mathcal{P},\mathcal{L}} - A_{[\mu}{}^{\mathcal{N}\rangle}A_{\nu}{}^{\mathcal{L}}A_{\rho]}{}^{\mathcal{P}}\right) \,. \tag{2.34}$$

Let us stress that G is not a gauge-covariant field strength. For instance, ignoring the scalar potential and its variation for the moment, one finds that the left-hand side of (2.33) varies under $\Lambda^{\mathcal{M}}$ as

$$\delta_{\Lambda}\left(\varepsilon^{\mu\nu\rho}G_{\mu\nu\rho}{}^{\mathcal{M}\mathcal{N}} + 2eA_{\mu}{}^{\langle\mathcal{M}}J^{\mu\mathcal{N}\rangle}\right) = -2\Lambda^{\langle\mathcal{M}}D_{\mu}\left(eJ^{\mu\mathcal{N}\rangle}\right) \tag{2.35}$$
$$+ g\varepsilon^{\mu\nu\rho}f^{\mathcal{K}\langle\mathcal{M}}{}_{\mathcal{P}}A_{\mu}{}^{\mathcal{N}\rangle}\Theta_{\mathcal{K}\mathcal{L}}\left(F_{\nu\rho}{}^{\mathcal{L}} + \tilde{J}_{\nu\rho}{}^{\mathcal{L}}\right)\Lambda^{\mathcal{P}} \,,$$

i.e. it rotates into the scalar equations of motion and the duality relation. In other words, despite the fact that G does not transform 'covariantly', the entire set of bosonic field equations is gauge-invariant. This concludes our discussion about three-dimensional gauged supergravity.
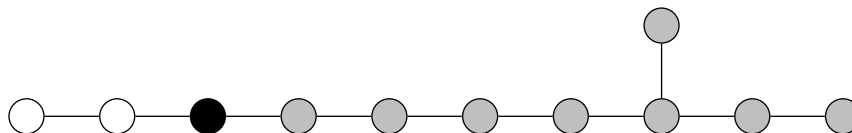
Figure 1: E_{11} decomposed under $SL(3,\mathbb{R}) \times E_{8(8)}$. The white nodes represent $SL(3,\mathbb{R})$, the gray nodes $E_{8(8)}$, and the black node is 'disabled'.

3. E_{11} and extended ungauged supergravity

In this section we are going to make the correspondence between ungauged supergravity and the Kac-Moody algebra E_{11} more precise. *A priori* there is a puzzle here since the $\Theta = 0$ limit of gauged supergravity leads to an ungauged theory in which the deformation and top-form potentials have disappeared from the Lagrangian. On the other hand, these same potentials are contained in the level decomposition of E_{11}. In this section we will show that a specific extended ungauged limit of gauged supergravity exists whose symmetries on all p–form potentials ($p = 0, 1, 2, 3$) are in precise correspondence to the non-linearly realized symmetries of (a truncation of) E_{11}, and which still contains all forms up to the top-form potentials. In the next subsection we first discuss the non-linear realization of E_{11}. In the following subsection we will discuss how the same result can be obtained by taking a limit of gauged supergravity.

3.1 Non-linear realization of E_{11}

We first consider the non-linear realization of E_{11}. In the case at hand we have to perform a level decomposition with respect to $SL(3,\mathbb{R}) \times E_{8(8)}$ (see figure 1), which are the space-time and duality subgroups. We restrict to the p–form algebra, which means that we truncate to generators that are totally antisymmetric in their 'space-time' indices μ, ν, ρ [15]. Specifically, this gives rise to generators $X^{\mu}{}_{\mathcal{M}}$, $Y^{\mu\nu}{}_{\mathcal{MN}}$, and $Z^{\mu\nu\rho}{}_{\mathcal{MN},\mathcal{P}}$ at level 1, 2 and 3, whose representations are given in table 1 [14]. We note that the level 2 generator is in precise correspondence with the linear constraint found for gauged supergravity, while the level 3 generator is consistent with the quadratic constraint. However, E_{11} allows for an additional top-form in **248**, which is not related to a quadratic constraint.[8] Here, these will not be considered further, and by abuse of notation we will denote the generator in which this additional **248** has been projected out also by $Z^{\mu\nu\rho}{}_{\mathcal{MN},\mathcal{P}}$. The non-trivial Lie brackets read

$$[X^{\mu}{}_{\mathcal{M}}, X^{\nu}{}_{\mathcal{N}}] = 2Y^{\mu\nu}{}_{\mathcal{MN}} , \tag{3.1}$$
$$[Y^{\mu\nu}{}_{\mathcal{MN}}, X^{\rho}{}_{\mathcal{P}}] = 3Z^{\mu\nu\rho}{}_{\mathcal{MN},\mathcal{P}} .$$

In order to determine the non-linearly realized E_{11} symmetry in this truncation, we have to introduce a group valued coset representative,

$$\mathcal{V} = \exp\left(A_{\mu}{}^{\mathcal{M}} X^{\mu}{}_{\mathcal{M}} + B_{\mu\nu}{}^{\mathcal{MN}} Y^{\mu\nu}{}_{\mathcal{MN}} + C_{\mu\nu\rho}{}^{\mathcal{MN},\mathcal{P}} Z^{\mu\nu\rho}{}_{\mathcal{MN},\mathcal{P}}\right) . \tag{3.2}$$

---

[8]Such top-forms could be related to space-time filling branes. Similar appearances of extra top-forms have been encountered in $D = 9, 10$ [15, 14].

Level	$SL(3,\mathbb{R}) \times E_{8(8)}$ representation	Generator
1	$(\mathbf{3}, \mathbf{248})$	$X^{\mu}{}_{\mathcal{M}}$
2	$(\bar{\mathbf{3}}, \mathbf{1} \oplus \mathbf{3875})$	$Y^{\mu\nu}{}_{\mathcal{MN}}$
3	$(\mathbf{1}, \mathbf{248} \oplus \mathbf{3875} \oplus \mathbf{147250})$	$Z^{\mu\nu\rho}{}_{\mathcal{MN},\mathcal{P}}$

Table 1: $SL(3,\mathbb{R}) \times E_{8(8)}$ representations within E_{11} up to level 3, of which the $SL(3,\mathbb{R})$ part is totally antisymmetric.

Here we have chosen the Borel gauge, in which only positive level generators enter. The action of the rigid symmetry group is given by

$$\mathcal{V} \to g\mathcal{V}h^{-1}(x) , \qquad g \in E_{11} , \tag{3.3}$$

where $h(x)$ denotes a local transformation which, if necessary, restores the chosen gauge for $\mathcal{V}$. However, after the gauge-fixing to positive levels in (3.2), it is sufficient for our purpose to consider the symmetry action by a group element truncated to positive level as well,

$$g = \exp\left(\Lambda_{\mu}{}^{\mathcal{M}} X^{\mu}{}_{\mathcal{M}} + \Lambda_{\mu\nu}{}^{\mathcal{MN}} Y^{\mu\nu}{}_{\mathcal{MN}} + \Lambda_{\mu\nu\rho}{}^{\mathcal{MN},\mathcal{P}} Z^{\mu\nu\rho}{}_{\mathcal{MN},\mathcal{P}}\right) . \tag{3.4}$$

Consequently, a compensating local transformation is not required. Acting with (3.4) on the coset representative (3.2), yields by use of the Baker-Campbell-Hausdorff formula and the Lie algebra (3.1) the following global symmetry transformations

$$
\begin{aligned}
\delta A_{\mu}{}^{\mathcal{M}} &= \Lambda_{\mu}{}^{\mathcal{M}} , \\
\delta B_{\mu\nu}{}^{\mathcal{MN}} &= \Lambda_{\mu\nu}{}^{\mathcal{MN}} + \Lambda_{[\mu}{}^{\langle\mathcal{M}} A_{\nu]}{}^{\mathcal{N}\rangle} , \\
\delta C_{\mu\nu\rho}{}^{\mathcal{MN},\mathcal{P}} &= \Lambda_{\mu\nu\rho}{}^{\mathcal{MN},\mathcal{P}} - \frac{3}{2} B_{[\mu\nu}{}^{\langle\mathcal{MN}} \Lambda_{\rho]}{}^{\mathcal{P}\rangle} + \frac{3}{2} \Lambda_{\mu\nu}{}^{\langle\mathcal{MN}} A_{\rho}{}^{\mathcal{P}\rangle} - \frac{1}{2} A_{[\mu}{}^{\langle\mathcal{M}} \Lambda_{\nu}{}^{\mathcal{N}} A_{\rho]}{}^{\mathcal{P}\rangle} .
\end{aligned}
\tag{3.5}
$$

In the next section we will show that these symmetries can also be obtained in a specific limit of supergravity.

3.2 Extended ungauged supergravity

In order to see the symmetry (3.5) in supergravity one has to consider a special ungauged limit. More precisely, taking the standard limit to ungauged supergravity, $g \to 0$, is equivalent to setting the embedding tensor to zero. This in turn eliminates the 1-, 2- and 3-forms from the action and, consequently, makes the comparison with E_{11} problematic. Moreover, from (2.28) one infers that in this naive limit scalar-dependent terms survive in the transformation rules as, for instance, $\delta_{\Lambda} B_{\mu\nu}{}^{\mathcal{MN}} = -\Lambda^{\langle\mathcal{M}} \tilde{J}_{\mu\nu}{}^{\mathcal{N}\rangle}$. These are not predicted by E_{11}, and so one has to take a more subtle limit. To be concrete, we first perform the following rescaling of the fields,

$$
\begin{aligned}
A_{\mu}{}^{\mathcal{M}} &\to g^{1/2} A_{\mu}{}^{\mathcal{M}} , \\
B_{\mu\nu}{}^{\mathcal{MN}} &\to g B_{\mu\nu}{}^{\mathcal{MN}} , \\
C_{\mu\nu\rho}{}^{\mathcal{MN},\mathcal{P}} &\to g^{3/2} C_{\mu\nu\rho}{}^{\mathcal{MN},\mathcal{P}} ,
\end{aligned}
\tag{3.6}
$$

and then take the limit $g \to 0$. This yields the Lagrangian,

$$\mathcal{L} \;=\; \mathcal{L}_0 - \frac{1}{4}\varepsilon^{\mu\nu\rho}\Theta_{\mathcal{MN}}G^{(0)}_{\mu\nu\rho}{}^{\mathcal{MN}} \; , \tag{3.7}$$

where $\mathcal{L}_0$ denotes the standard Lagrangian of ungauged supergravity. Here, $G^{(0)}_{\mu\nu\rho}{}^{\mathcal{MN}}$ is the $g \to 0$ limit of $G_{\mu\nu\rho}{}^{\mathcal{MN}}$, given by

$$G^{(0)}_{\mu\nu\rho}{}^{\mathcal{MN}} \;=\; \partial_{[\mu}B_{\nu\rho]}{}^{\mathcal{MN}} + A_{[\mu}{}^{\langle\mathcal{M}}\partial_\nu A_{\rho]}{}^{\mathcal{N}\rangle} \; . \tag{3.8}$$

We note that, in contrast to the gauged expression in (2.34), this represents a gauge-invariant field strength. The Lagrangian (3.7) is equivalent to standard ungauged supergravity in that it merely represents an extension by topological 1- and 2-forms with vanishing curvatures.[9] To be more precise, the embedding tensor now acts as a Lagrange multiplier that sets the curvature of the 2-form to zero, while the field equations for $A_\mu{}^{\mathcal{M}}$ imply that their (abelian) field strengths vanish.

Let us now turn to the symmetries that survive in this limit. Rescaling the symmetry parameters as for the fields in (3.6), i.e. $\Lambda^{\mathcal{M}} \to g^{1/2}\Lambda^{\mathcal{M}}$, etc., yields the following limit of the gauge symmetries (2.28),

$$\begin{aligned}
\delta_\Lambda A_\mu{}^{\mathcal{M}} &= \partial_\mu \Lambda^{\mathcal{M}} \; , \\
\delta_\Lambda B_{\mu\nu}{}^{\mathcal{MN}} &= \partial_{[\mu}\Lambda_{\nu]}{}^{\mathcal{MN}} + \partial_{[\mu}\Lambda^{\langle\mathcal{M}}A_{\nu]}{}^{\mathcal{N}\rangle} \; , \\
\delta_\Lambda \hat{C}_{\mu\nu\rho}{}^{\mathcal{MN},\mathcal{P}} &= \partial_{[\mu}\Lambda_{\nu\rho]}{}^{\mathcal{MN},\mathcal{P}} - \frac{3}{2}\partial_{[\mu}\Lambda^{\langle\mathcal{P}}B_{\nu\rho]}{}^{\mathcal{MN}\rangle} + \frac{3}{2}\partial_{[\mu}\Lambda_\nu{}^{\langle\mathcal{MN}}A_{\rho]}{}^{\mathcal{P}\rangle} - \frac{1}{2}A_{[\mu}{}^{\langle\mathcal{P}}A_\nu{}^{\mathcal{M}}\partial_{\rho]}\Lambda^{\mathcal{N}\rangle} \; .
\end{aligned} \tag{3.9}$$

Here we performed the field redefinition

$$\hat{C}_{\mu\nu\rho}{}^{\mathcal{MN},\mathcal{P}} \;=\; C_{\mu\nu\rho}{}^{\mathcal{MN},\mathcal{P}} + \frac{3}{2}A_{[\mu}{}^{\langle\mathcal{P}}B_{\nu\rho]}{}^{\mathcal{MN}\rangle} \; . \tag{3.10}$$

In particular we observe that the scalar-dependent terms drop out. Specifying the gauge parameters to linear space-time dependence according to

$$\Lambda^{\mathcal{M}} \;=\; x^\rho \Lambda_\rho{}^{\mathcal{M}} \; , \qquad \Lambda_\mu{}^{\mathcal{MN}} \;=\; x^\rho \Lambda_{\rho\mu}{}^{\mathcal{MN}} \; , \qquad \Lambda_{\mu\nu}{}^{\mathcal{MN},\mathcal{P}} \;=\; x^\rho \Lambda_{\rho\mu\nu}{}^{\mathcal{MN},\mathcal{P}} \; , \tag{3.11}$$

gives precisely the global symmetry in (3.5) predicted by E_{11}.

We note that in the $g \to 0$ limit the top-form vanishes from the Lagrangian but does have a well-defined gauge transformation rule which is in accordance with the E_{11} algebra. Therefore only the (truncated) E_{10} subalgebra is non-trivially realized at the level of the Lagrangian. Finally, in the $g \to 0$ limit, the gauge algebra closes *off-shell*, as it should be since it matches the E_{11} results, which a priori do not contain information about the equations of motion.

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[9]Recently, a similar use of topological fields in the context of the Kac-Moody approach has been made in [35].

4. Conclusions

In this note we compared a level decomposition based on the very extended Kac-Moody algebra E_{11} with a particular limit of maximal three-dimensional gauged supergravity. Before taking the limit, the gauged supergravity theory contains besides scalars and vectors also deformation and top-form potentials on which the gauge algebra, which we determined explicitly, closes *on-shell*. After taking the limit we are left with a Lagrangian containing scalars, vectors and deformation potentials on which the gauge algebra closes *off-shell*. This gauge algebra allows for a rigid truncation, which in turn realizes an E_{10} subalgebra of E_{11}. To obtain the full E_{11} prediction one must include the top-form potentials which, however, do not occur in the Lagrangian.[10] It is intriguing to note that the lowest-order terms in the variation δC of the top-form as predicted by E_{11} are, from the supergravity side, required for canceling the *higher-order* terms in Θ in the variation of the action. So in this sense, E_{11} does know about the gauging.

It is natural to expect that the need for a rescaling in order to match the E_{11} prediction for the deformation and top-form potentials appears in any dimension. In particular, it would be interesting to verify this in the case of $D = 5$ analyzed in [28]. However, there the full gauge transformations have been given up to the 3-forms, for which a rescaling is not required. Thus, a comparison with our results must await an exhaustive analysis of the 4- and 5-forms in $D = 5$.

Moreover, it would be interesting to extend, for three dimensions, the relation between extended ungauged supergravity and E_{10} and/or E_{11} to the gauged case. Since, in going from the ungauged to the gauged case, the closure of the gauge algebra goes from off-shell to on-shell we expect that dynamics will play a non-trivial role in this extension. Recently, for the case of E_{11}, a proposal for such a relationship in the gauged case has been made [28]. It would be interesting to see wether this proposal yields the details and in particular the on-shell closure of the three-dimensional gauge algebra. Since dynamics is involved it would be interesting to also consider the relationship from the point of view of the E_{10} coset model [3, 2, 4] where dynamics is naturally included via the sigma model equations of motion. This would extend the analysis of [33] for $D = 10$ massive supergravity to a case where the gauging of a symmetry is involved. We hope to report on the results of such an investigation in the nearby future [36].

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